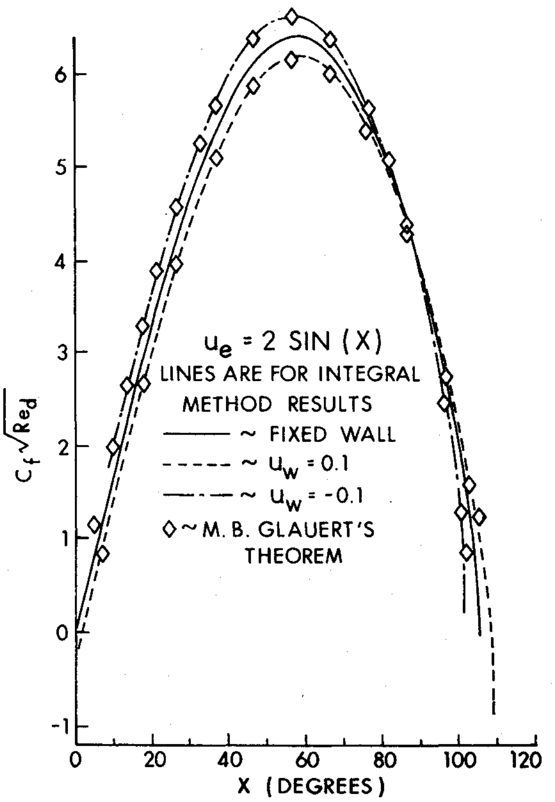

Fig. 2 Comparison with Glauert's result for a cylinder.

were so different from each other that interpolating between their integral values did not yield good results. As integration proceeds to smaller values of u_w/u_e (downstream of $u_w=u_e$) the momentum and displacement thickness quickly converge back to the correct level with the skin friction being somewhat slower to converge. Thus, the integral boundary-layer calculation of Rott's solution is shown to be stable, so that after the interval of inaccurate calculation, the integral quantities converge rapidly back to Rott's[5] solution. For the application to rotating cylinders, the region of inaccuracy did not affect the results significantly. However, for some other possible applications, the similarity profiles might be augmented in the region of less accuracy by some suitably designed profiles satisfying some compatibility conditions.

A comparison of the method can also be made with results obtained by a theorem of Glauert.[6] This theorem can be used to estimate moving-wall boundary-layer skin friction for the case of a slowly rotating cylinder. The result of the theorem applied to rotating cylinders is the following: if one knows the pressure distribution on a rotating cylinder and the boundary-layer development on the equivalent fixed cylinder, the boundary-layer development on the slowly rotating cylinder may be calculated approximately. For the slowly rotating cylinder a result obtained by Glauert[6] is that

$$\tau \simeq [\tau_l^2 - 2\bar{u}_w \mu\rho\bar{u}_e (\mathrm{d}\bar{u}_e/\mathrm{d}\bar{x})]^{1/2} \tag{6}$$

where τ is the shear stress for the moving wall and τ_l is the shear stress for the fixed wall. Here the subscript 1 denotes the fixed wall conditions. Equation (6) can be nondimensionalized for this particular problem with the result:

$$c_f(Re_d)^{1/2} \simeq [(c_f)_l^2 Re_d - 32u_w \sin(2x)]^{1/2} \tag{7}$$

The skin friction results obtained by the integral method for cylinder rotation rates of -0.1, 0.0, and 0.1 are shown in Fig. 2 as a function of position on the cylinder. The skin friction results using Glauert's theorem are included for comparison. The nondimensional velocity distribution is 2 sin$x$. The results obtained by Glauert's[6] theorem for slowly rotating cylinders agree well with the results obtained by the integral technique except near separation where Glauert's approximation tends to break down.

IV. Conclusions

A two-equation integral technique for calculating the incompressible two-dimensional boundary-layer development on a moving wall has been developed. By comparison with Rott's[5] solution and a result obtained by Glauert's[6] theorem, it has been determined that accurate calculations of boundary-layer development can be obtained over the major portion of a slowly rotating cylinder. For the upstream moving wall, the lower limit for accurate stable integration of the boundary is found to be $u_w/u_e = -0.3$ and a lower limit for u_w/u_e is to be expected from the parabolic nature of the boundary-layer equations. It appears that with the integral technique, larger reverse flows can be handled than with finite-difference methods.[7,8] The flows used with the two techniques were different, however, and so the lack of a direct comparison prevents complete confirmation.

References

[1] Fansler, K.S., "Integral Analysis of Boundary Layer on Moving Wall and Its Application to a Spinning Cylinder in Crossflow," Ph.D. Thesis, May 1974, University of Delaware, Newark, Del.

[2] Danberg, J.E. and Fansler, K.S., "Similarity Solutions of the Boundary-Layer Equations for Flow Over a Moving Wall," Rept. 1714, April 1974, Ballistic Research Laboratories, Aberdeen Proving Ground, Md.

[3] Lees, L. and Reeves, B., "Supersonic Separated and Reattaching Laminar Flows: I General Theory and Application to Adiabatic Boundary Layer/Shockwave Interaction," *AIAA Journal,* Vol. 2, Nov. 1964, pp. 1907-1920.

[4] Reeves, B. and Lees, L., "Theory of Laminar Near Wake of Blunt Bodies in Hypersonic Flow," *AIAA Journal,* Vol. 3, Nov. 1965, pp. 2061-2074.

[5] Rott, N., "Unsteady Viscous Flow in the Vicinity of a Stagnation Point," *Quarterly of Applied Mathematics,* Vol. 13, 1956, pp. 444-451.

[6] Glauert, M.B., "A Boundary Layer Theorem, with Applications to Rotating Cylinders," *Journal of Fluid Mechanics,* Vol. 2, 1957, pp. 89-99.

[7] Catherall, D. and Mangler, K.W., "The Integration of the Two-Dimensional Laminar Boundary-Layer Equations Past the Point of Vanishing Skin Friction," *Journal of Fluid Mechanics,* Vol. 26, Part 1, 1966, pp. 163-182.

[8] Carter, J.E. and Wornom, S.F., "Forward Marching Procedure for Separted Boundary-Layer Flows," *AIAA Journal,* Vol. 13, Aug. 1975, pp. 1101-1103.

[9] Rosenhead, L. (Ed.), *Laminar Boundary Layers,* Oxford University Press, Oxford, 1963, pp. 292-311.

[10] Libby, P.A. and Liu, T.M., "Further Solutions of the Falkner-Skan Equation," *AIAA Journal,* Vol. 5, May 1967, pp. 1040-1042.

Rapid Analysis of Damaged Structure

Harold Switzky*
Fairchild Republic Company, Farmingdale, N.Y.

AN iteration algorithm based upon known mathematical principles is presented for comment and exploitation. The matrix operations and decisions can be performed very rapidly in any digital computer.

Digital procedures for the analysis of a loaded structure are readily available. Unfortunately, the programs require the solution of large numbers of simultaneous equations which

Received Sept. 11, 1975; revision received March 8, 1976.
Index categories: Structural Static Analysis; Structural Stability Analysis.

*Structures/Material Staff. Associate Fellow AIAA.

consume a large deal of computer time. Thus, the reanalysis of a structure for several possible damages (or modifications) is extremely costly, and can be performed more economically employing an iterative scheme. In fact, if the number of the simultaneous equations becomes critically large, then even the original analysis can be performed best by an iterative scheme. The cost efficiency of the solutions are dependent upon the particular iterative scheme, some being more efficient than others. The iterative scheme presented is very efficient because it utilizes a procedure to accelerate convergence and reduce the number of trial solutions, as well as employing only rapid mathematical operations. The technique, although directly applicable to linear structure, can be of value in nonlinear problems, provided the nonlinearity is limited to the finite elements.

The stiffness (K) procedure, readily available in digital libraries (e.g., NASTRAN, MAGIC, etc.), is first employed to analyze the unmodified structure for the displacement (Δ_0) due to the applied loads (P).

i.e. $$P = K\Delta_0 \qquad (1a)$$

and $$\Delta_0 = K^{-1} P = FP \qquad (1b)$$

Let us now modify or damage the structure (δK) and try to determine the new ($\Delta_0 + \delta\Delta$) displacements.

$$P = (K - \delta K) \quad (\Delta_0 + \delta\Delta) \qquad (2a)$$

$$P = K\Delta_0 + K\delta\Delta - \delta K\Delta_0 - \delta K\delta\Delta \qquad (2b)$$

Subtracting Eq. (1a) from Eq. (2b) and rearranging terms results in the solution

$$\delta\Delta = (K - \delta K)^{-1}\delta K\Delta_0 = (I - K^{-1}\delta K)^{-1} K^{-1}\delta K\delta_0 \qquad (2c)$$

$$\therefore \delta\Delta = \left[\sum_{n=0}^{\infty} A^n\right] A\Delta_0 = \left[\sum_{n=0}^{\infty} A^n\right]\delta\Delta_0 \qquad (2d)$$

where $$A = K^{-1}\delta K \qquad (2e)$$

Note that the solution is expressible as an infinite geometric matrix series. Each successive term of the series represents the elastic deformation of the unmodified structure, due to the difference between the applied load and the internal load, generated by the intermediate displacement of the modified structure. The series converges only if the matrix A has a norm of less than unity. The norm is analogous to the geometric ratio of a series in establishing convergence of the series. The norm is a numerical measure of the matrix and can be defined satisfactorily (for this application) as the magnitude of the largest (first) eigenvalue (λ) of the matrix. This norm is less than the trace and greater than the determinant for positive definite matrices. The deformation change matrix A generally has a norm of less than unity (due to a loss of stiffness (δK)) provided the remaining structure is stable. It can be shown readily that the norm is less than one if the first eigenvectors of the modified and unmodified structures are quite close, and that the first eigenvalue of the modified structure is less than twice that of the original structure.

Iterations of higher powers of the matrix, operating upon any vector with a component of the first eigenvector (ξ_1), will converge to a vector proportional to the first eigenvector with each iteration resulting in a vector which is approximately the eigenvalue times the previous vector.

i.e., $$A^{n+1}\Delta_0 = A(A^n\Delta_0) \sim \lambda A^n\Delta_0 \sim \lambda^{n+1}\xi_1 \qquad (3)$$

This is due to the fact that multiplication of a vector by a matrix results in a rotation and change in magnitude of the vector. Each eigenvector component is modified in direct proportion to the corresponding eigenvalue. Thus, multiplication by a matrix raised to a high power will result in a vector which is primarily in the highest eigenvalue mode. The geometric series solution is even more biased to the highest eigenvalue mode since ($1/1-\lambda$) is also highest for this mode. Thus, the solution of Eq. (2a) is very close to the first few terms of the series plus a function proportional to $\xi_1/(1-\lambda)$, as indicated in the following. Let us rewrite Eq. (2) as a recursive equation

i.e. $$\delta\Delta = K^{-1}\delta K\Delta_0 + K^{-1}\delta K\delta\Delta$$

and

$$\delta\Delta_{(n+1)} = A\Delta_0 + A\delta\Delta_{(n)} = \delta\Delta_0 + A\delta\Delta_{(n)} \qquad (4a)$$

$$= \delta\Delta_0 + A[\delta\Delta_0 + A\delta\Delta_{(n-1)}] = \delta\Delta_0 + A\delta\Delta_0 + A^2\delta\Delta_{(n-1)} \qquad (4b)$$

or

$$\delta\Delta_{n+1} = (I + A + A^2 + \ldots + A^n + A^{n+1})(\delta\Delta_0 = A\Delta_0) \qquad (4c)$$

Equation (4c) is the finite approximation to the solution presented in Eq. (2d). If we define the difference between successive solutions ($\delta^2\Delta$) and utilize the approximation expressed by Eq. (3), we obtain

$$\delta^2\Delta_{(n+1)} = \delta\Delta_{(n+1)} - \delta\Delta_{(n)} = A^{n+1}\delta\Delta_0 \sim \lambda A^n\delta\Delta_0 = \lambda\delta^2\Delta_{(n)} \qquad (5a)$$

and $$\delta\Delta \sim \delta\Delta_{(n)} + \frac{\delta^2\Delta_{(n+1)}}{1-\lambda} \qquad (5b)$$

The last term is an approximation of the remaining terms of the infinite series, and becomes more accurate with increasing n. The error in the approximation is directly proportional to the deviation of $\delta^2\Delta_{(n+1)}$ from the first eigenvector.

A readily determinable estimate of the eigenvalue λ is obtained from

$$\frac{E_{n+1}}{E_n} = \frac{[\delta^2\Delta_{(n+1)}]^T[\delta^2\Delta_{n+1})]}{[\delta^2\Delta_{(n)}]^T[\delta^2\Delta_{(n)}]} \sim \frac{\lambda^2[A^n\delta\Delta_0]^T[A^n\delta\Delta_0]}{[A^n\delta\Delta_0]^T[A^n\delta A_0]} = \lambda^2 \qquad (5c)$$

where $(\delta^2\Delta)^T$ is the transpose of ($\delta^2\Delta$). An iterative scheme based upon Eq. (5) results in more rapid solutions of Eq. (2) than would be obtained by the iterative scheme of Eq. (4), since the former scheme permits extrapolation to larger increments with a smaller number of trial solutions.

To establish an engineering solution in a reasonable time, it is necessary to establish criteria of acceptable accuracy (nearness) of the trial solution to the true solution. The

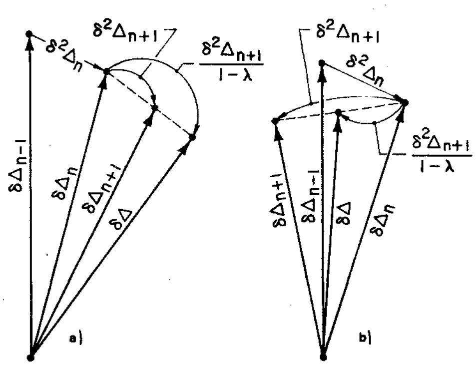

Fig. 1 Vectorial representation of geometric solution: a) nonoscillating convergence; b) oscillating convergence.

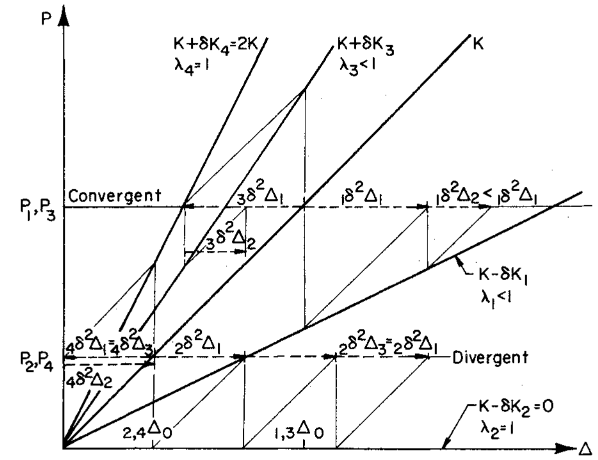

Fig. 2 Convergence modes $|_i\delta^2\Delta_{n+1}| = \lambda_i\,|_i\delta^2\Delta_n|$.

simultaneous satisfaction of two criteria is recommended for terminating the iteration cycles. One criteria is that the change in deflections between two iterations ($\delta^2\Delta_{(n)}$), as measured by E_n, be sufficiently small. The other criteria is that the change in strain energy (δU) between two iterations be sufficiently small as compared to the strain energy (U) to assure that the stress changes are not of any engineering significance.

i.e. $E_n \leq \text{TOL } 1 \sim$ (maximum acceptable deflection error)^2 times the number of displacements (6a)

$$\frac{\delta U}{U} \leq \text{TOL } 2 \sim (\text{maximum acceptable stress error ratio})^2 \tag{6b}$$

where

$$\delta U = (P)^T(\delta^2\Delta) \tag{6c}$$

and

$$U = (P)^T(\Delta_0 + \delta\Delta) \tag{6d}$$

It should be noted that the recommended iterative scheme is even more efficient when the modification increases the stiffnesses of the members. In this case the geometric matrix series solution represents an alternating series. This requires a large number of linear (oscillating) iterations to approach the solution, while the geometric series extrapolation requires a much smaller number of trial solutions. The recommended geometric solution is

$$\delta\Delta = \delta\Delta_n + \frac{\delta\Delta_{n+1} - \delta\Delta_n}{1 - (\text{sgn}[\delta^2\Delta_{n+1}]^T[\delta^2\Delta_n])(E_{n+1}/E_n)^{1/2}} \tag{7}$$

The iterative algorithm consists of selecting a good initial guess as to the change in displacement ($\delta\Delta_{n-1}$) due to the structural modification (δK). The first initial guess is $\delta\Delta_0 = A\Delta_0$. Equation (4a) is then employed to determine values of $\delta\Delta_n$ and $\delta\Delta_{n+1}$. The values of $\delta^2\Delta_n$, $\delta^2\Delta_{n+1}$, E_n, E_{n+1}, and λ are also calculated from Eqs. (5a) and (5c) and substituted into Eq. (7) to determine a new initial guess. This computational scheme is repeated as many times as required. The decision as to the acceptable accuracy of the last initial guess is based upon satisfaction of Eq. (6). It should be noted that the iterative scheme does not require the inversion of the stiffness matrix (K^{-1}). Any stiffness such as the main diagonal or upper triangle, which are easily inverted (F), may be utilized provided the norm of $F\delta K$ is less than one.

The mode of convergence is schematically presented in Figs. 1 and 2. Figure 2 represents the Eq. (2d) solution of a one degree of freedom (invariant eigenvector) structure. Note that the value of $\delta^2\Delta_n$ always decrease whenever $\lambda = |\delta K/K| < 1.0$, guaranteeing convergence. A reduction in stiffness results in positive $\delta^2\Delta_n$, while an increase in stiffness causes an oscillation in sign and a slower convergence. Both solutions would be accelerated with the algorithm. Values of $\lambda \geq 1.0$ result in nondecreasing values of $\delta^2\Delta_n$, indicating that Eqs. (2e) or (5b) should not be utilized for major increases of stiffness or an unstable structure. The increase of stiffness can be handled by an arbitrary magnification of the original stiffness ($K_2 = \mu K_1$), so as to result in $\lambda_{12} = \lambda_{11}/\mu < 1.0$.

The algorithm assumes that the linear damaged structure has adequate strength to withstand the resulting calculated stresses, which are a known linear transformation of the displacements. If this is not justified, then the linear structure can be reanalyzed with a "pseudo load" (equal to the load capacity of the overstressed members, e.g., yield, buckling, fracture) and an incremental δK representing the elastic stiffness of these members. The iterative algorithm applies to linear structure, but can be advantageously reiterated for nonlinear structures, provided that the effect of the nonlinearity upon the δK can be evaluated in the manner indicated previously.

Note that efficient iterative solutions of original analysis problems of large number of degrees of freedom can be obtained by employing the readily available inverses of the main diagonal, without or with the upper triangle, as the flexibility matrix (F). The difference between these matrices and the total stiffness matrix (K) can be set equal to the change in stiffness (δK). The latter technique (Gauss-Siedel) is more efficient since the norm of the corresponding change matrix (A) is smaller. Utilizing the modification represented by extrapolation to the geometric series approximation should significantly increase the efficiency of these oscillating iterative techniques. The geometric series algorithm can also be utilized to increase the efficiency for the determination of eigenvalues and eigenvectors by iterative techniques.

Slow Ions in Plasma Wind Tunnels

W. A. Oran*
NASA Marshall Space Flight Center, Huntsville, Ala.
U. Samir†
University of Michigan, Ann Arbor, Mich.
Tel Aviv University, Ramat Aviv, Israel
and
N. H. Stone‡
NASA Marshall Space Flight Center, Huntsville, Ala.

Introduction

THE interaction between a satellite and its space environment, which created disturbed zones around the satellite, has been studied since the early 1960's. Numerous theoretical models which describe aspects of the interaction are now available (e.g., Refs. 1-3); although, the degree of applicability of these models is not always well understood.

Received February 2, 1976; revision received April 19, 1976.
Index categories: Rarefied Flows; Plasma Dynamics and MHD; Research Facilities and Instrumentation.

*Research Physicist, Magnetospheric and Plasma Physics Branch, Space Sciences Laboratory.
†Research Scientist, Department of Atmospheric and Oceanic Sciences, University of Michigan; Associate Professor, Department of Geophysics and Planetary Science, Tel Aviv University.
‡Research Physicist, Magnetospheric and Plasma Physics Branch, Space Sciences Laboratory.